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Steady flows drawn from a stably stratified reservoir

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Perfect-fluid theory is applied to the description of steady motions that can be generated as the outflow into a horizontal channel from a large reservoir of incompressible heavy fluid whose density is an arbitrary decreasing function of height. A particular aim is to pinpoint the significance of an already known class of flows, called *self similar*, which satisfy the approximate (shallow-water) equations applicable when the horizontal scale of the motion greatly exceeds its vertical scale, but which have not until now been shown to match the downstream conditions that primarily determine the motion in practice.

New variational principles are introduced characterizing the class of self-similar flows: in §2 there is a characterization in terms of flow force among parallel flows realized asymptotically in a uniform channel, in §3 among a wider range of possibilities including periodic flows, and in §6 among supercritical flows realized in a convergent-divergent channel. Aspects of general flows in channels of gradually varying breadth are treated in §§4 and 5, including the remarkable fact, proven in §5, that every steady flow outside but close to the self-similar class must somewhere undergo a local crisis unaccountable by the shallow-water approximation. Practical interpretations afforded by the theoretical results are noted in §7.

1. Introduction

It has been known for some time that the equations describing the steady, gradually converging flow of a stably stratified perfect fluid from a large reservoir into a horizontal channel have a simple solution, according to which the stream-surfaces duplicate those in a corresponding open-channel flow of a homogeneous fluid. Wood (1968) appears to have been the first to put this fact on record, although evidently it was then already known to others. For the special flows in question he introduced the term *self similar*, emphasizing their property that the heights of stream-surfaces are everywhere in the same ratio, and this useful term will be readopted here. These flows were also demonstrated by Yih (1969), whose theoretical description went a little further than Wood's and included an appraisal of the shallow-water approximations that underlie the mathematical model. Yih's account was properly cautious, moreover, about the physical significance attributable to the special class of flows. Other steady flows were recognized to be possible from any given reservoir that either has multiple discrete layers or is continuously stratified, and it was noted that without evidence of the special flows satisfying terminal conditions that can be imposed downstream, there is no *a priori* reason for their ever being realizable.

Various outstanding questions concerning the class of self-similar flows will be answered here, most notably the question of their realizability in practically significant situations. In the case of drainage from a layer in the reservoir bounded by a discontinuity in density, a self-similar flow will be shown to arise as the extreme state before hitherto stagnant fluid is drawn into motion; and in the case of selective withdrawal from a continuously stratified reservoir, self-similar flows will be shown always to have priority. The theoretical models studied are idealized, taking no account of effects due to viscosity or diffusion, but the conclusions therefrom should have bearing on certain natural flows on a large scale (cf. Wood 1968, § 1).

Items of theory are developed in §§ 2–6, and finally the powerful interpretations that they provide are assembled in § 7. A simple but very informative variational principle in terms of flow force (horizontal pressure force plus momentum flux) is established in § 2 for horizontal flows drawn from a reservoir into a straight channel, and the principle is extended in § 3 to include wavy flows. In § 4 the theory of steady flows in a channel of gradually varying breadth is reviewed, the self-similar solution of the approximate hydrodynamic problem is noted, and some difficult questions posed by flows other than the self-similar one are defined. In § 5 it is proved that no steady flow neighbouring on the self-similar flow exists corresponding to a smooth solution of the shallow-water equations, and in § 6 a second, much less transparent variational principle is given relating to supercritical flows in the divergent part of a channel with a throat.

The concluding discussion in § 7 relies on arguments that generalize the following elementary idea from open-channel hydraulics. Suppose a stream of water having asymptotic depth h and velocity u is drawn from a large reservoir into a horizontal open channel of rectangular cross-section. The flow force of the stream is given by $S = \rho(u^2h + \frac{1}{2}gh^2)$, where ρ is the (uniform) density; and the Bernoulli law gives $u^2 = 2g(H-h)$, where H is the height of the free surface in the reservoir above the bottom of the channel. Thus one has $S = \rho g(2Hh - \frac{3}{2}h^2)$, which achieves a unique maximum value $S_m = \frac{2}{3}\rho g H^2$ when $h = \frac{2}{3}H$ and consequently $u^2 = gh$. If the reservoir is connected through a contraction, the flow can, of course, be controlled from the downstream end of the channel, say by lowering a weir or by operating a pump which takes up the water. In every case an adjustment increasing the steady outflow can be reckoned to induce an increase in the flow force of the oncoming stream, which has to match the greater rate of extraction of momentum from the system; but the preceding simple result shows that upon S being raised to the value S_m, no further increase can in any way be induced. It is impossible to realize a stream with $S > S_m$ by outflow from the given reservoir. This interpretation of the critical condition at which downstream control is lost may be appreciated to provide better physical insights, and to admit wider generalization, than the more usual one which focuses on the maximum of flow rate uh for a given H. An immediate advantage of it, generalized in § 3, is that the maximum principle for S easily extends to wavy flows, every one of which is known to realize a smaller flow-force value than a uniform flow with the same H (Benjamin & Lighthill 1954).

2. Steady flows into a uniform channel

As indicated in figure 1, a stably stratified fluid lying on a rigid horizontal plane is considered to flow from an infinitely wide reservoir into a straight channel, whose cross-section following a smooth entry region is rectangular and uniform. The motion is assumed to be steady, and the fluid to be inviscid, incompressible and non-diffusive, so that its density has a constant value on each stream-surface, varying only among

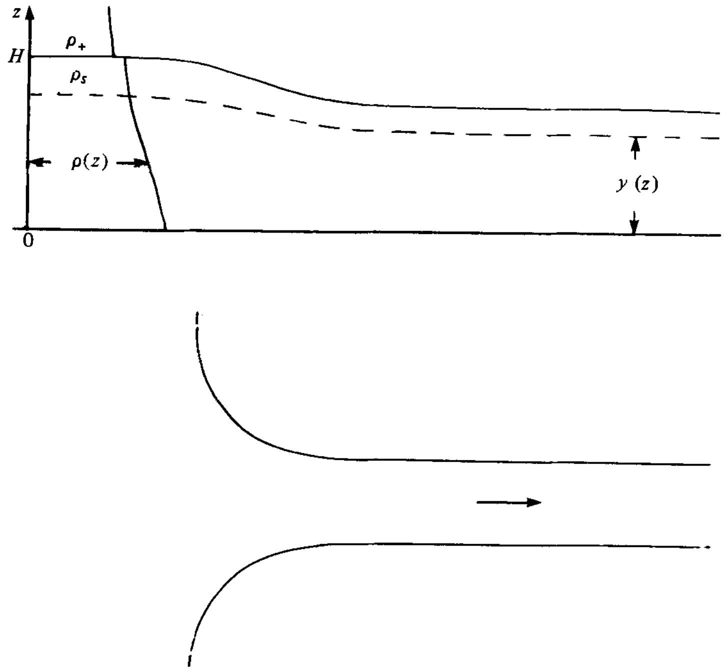

FIGURE 1. Illustration of flow from a stratified reservoir into a straight channel.

these surfaces. Thus, if z denotes their original heights above the bottom in the reservoir where the fluid is at rest, the density is representable everywhere by $\rho = \rho(z)$.

All pressures will be expressed relative to the pressure at the level $z = H$ in the reservoir, below which level the fluid is drawn into the channel and above which none flows when the steady motion is established. According to the hydrostatic law, the pressure at lower levels in the reservoir is given by

$$p = g\int_z^H \rho(\xi)\,d\xi,$$

and hence the total head of the fluid above the bottom by

$$R(z) = gz\rho + p = g\left\{z\rho + \int_z^H \rho(\xi)\,d\xi\right\} = g\left\{\rho_s H - \int_z^H \xi\rho'(\xi)\,d\xi\right\}, \tag{2.1}$$

where $\rho_s = \rho(H-)$. We also need the notation $\rho_+ = \rho(H+)$, supposing for now that $\rho_+ < \rho_s$ (see figure 1), but the case $\rho_+ = \rho_s$ will be covered. In (2.1), ρ' denotes $d\rho/dz$, being a non-positive function by the assumption of stability, and henceforth accents will be used only to denote derivatives of z-dependent functions.

If the flow approached asymptotically along the channel is horizontal (i.e. free from waves), the height y of the stream-surfaces in it is a function of z alone, and we write $y(H-) = h$. Since the fluid is at rest at heights above h, the layer between heights h and H must be filled with stagnant fluid of density ρ_+. Hence the pressure in the horizontally moving fluid is given, according to the hydrostatic law, by

$$p = g\left\{\rho_+(H-h)+\int_y^h \rho\, d\tilde{y}\right\} = g\left\{\rho_+(H-h)+\int_z^H \rho(\xi)\, y'(\xi)\, d\xi\right\}$$

$$= g\left\{\rho_+(H-h)+\rho_s h-\rho y-\int_z^H y\rho'\, d\xi\right\}. \quad (2.2)$$

If q is the magnitude of the velocity in the fluid, the Bernoulli law tells us that

$$R = gy\rho+p+\tfrac{1}{2}\rho q^2 \quad (2.3)$$

is constant on any stream surface, specified by the value of z; and in the asymptotic flow $q^2 = u^2$, the square of the single, horizontal component $u(z)$ of velocity. Hence, using the expressions (2.1) for $R(z)$ and (2.2) for p, we obtain

$$\rho u^2 = 2g\left\{\hat{\rho}(H-h)-\int_z^H (\xi-y)\,\rho'\, d\xi\right\}. \quad (2.4)$$

in which $\hat{\rho} = \rho_s-\rho_+$.

The key to subsequent physical interpretations is to consider the *flow force* S defined as the sum of horizontal pressure force and momentum flux, per unit span, in the whole layer of fluid affected by the steady flow (cf. Benjamin 1966, §2). Thus S is the integral of $p+\rho u^2$ with respect to height, from the bottom to the height H. The contribution to S from the pressure in the stationary fluid above the uppermost stream-surface $y = h$ is plainly $\tfrac{1}{2}g\rho_+(H-h)^2$, and hence

$$S = \tfrac{1}{2}g\rho_+(H-h)^2+\int_0^h (p+\rho u^2)\, dy = \tfrac{1}{2}g\rho_+(H-h)^2+\int_0^H (p+\rho u^2)\, y'\, dz. \quad (2.5)$$

For substitution in this integral, p can be expressed by (2.2) and the momentum flux density ρu^2 by (2.4) with $q^2 = u^2$. We thus obtain at once

$$S/g = \tfrac{1}{2}\rho_+(H-h)^2+\rho_+(H-h)\,h+\rho_s h^2+2\hat{\rho}(H-h)\,h$$

$$-\int_0^H\left\{\rho y-\int_z^H [2\xi-y(\xi)]\,\rho'(\xi)\, d\xi\right\} y'\, dz,$$

and after an integration by parts

$$S/g = \tfrac{1}{2}\rho_+ H^2+\hat{\rho}(2Hh-\tfrac{3}{2}h^2)+\int_0^H (-\rho')\,(2zy-\tfrac{3}{2}y^2)\, dz. \quad (2.6)$$

With regard to the terms of (2.6), note the identity

$$2zy-\tfrac{3}{2}y^2 = \tfrac{2}{3}z^2-\tfrac{1}{6}(2z-3y)^2,$$

including its instance with $z = H$, $y = h$. Note also, from the assumption of stable stratification in the reservoir, that $\hat{\rho} \geqslant 0$ and $-\rho'$ is a non-negative function on $[0, H)$. It therefore follows from (2.6) that S is an *absolute maximum* when

$$y = \tfrac{2}{3}z, \quad (2.7)$$

which case is the self-similar flow that is *critical* in a sense to be recalled presently.

The maximum value of S is

$$\left.\begin{aligned} S_m &= g\left\{\tfrac{1}{2}\rho_+ H^2 + \tfrac{2}{3}\hat{\rho}H^2 + \tfrac{2}{3}\int_0^H (-\rho')\,z^2\,dz\right\} = S_0 + \tfrac{1}{6}g\left\{\tfrac{1}{2}\rho_s H^2 + \int_0^H (-\rho')\,z^2\,dz\right\}, \\ &\text{where} \\ S_0 &= g\left\{\tfrac{1}{2}\rho_s H^2 + \tfrac{1}{2}\int_0^H (-\rho')\,z^2\,dz\right\} = g\int_0^H \rho z\,dz \end{aligned}\right\} \quad (2.8)$$

is the value of S for the fluid in a state of rest. This result generalizes the well-known principle recalled in § 1 concerning open-channel flows of a homogeneous fluid, a case that is recovered by putting $\rho_+ = 0$ and $\rho' \equiv 0$ in $[0, H)$.

For the self-similar flow described by (2.7), we have from (2.4) that

$$\rho u^2 = \tfrac{2}{3}g\left\{\hat{\rho}H - \int_z^H \rho' z\,dz\right\} = \tfrac{2}{3}g\{R(z) - R(H+)\} = g\left\{\hat{\rho}h - \int_y^h \frac{d\tilde{\rho}}{dy}y\,dy\right\}, \quad (2.9)$$

where $\tilde{\rho} = \tilde{\rho}(y) = \tilde{\rho}(\tfrac{2}{3}z)$ is written in place of $\rho(z)$. From the definition $u = d\psi/dy$, $\psi(0) = 0$, (2.9) can be used to find the stream-function $\psi(y)$ for this flow.

A convenient example

The example now introduced will be used later to illustrate other aspects of the problem. Let

$$\left.\begin{aligned} H = 1, \quad \rho(z) &= \rho_0(1-\beta z) \quad (0 < \beta < 1), \\ \rho_s = \rho_0(1-\beta) &= \rho_+ \quad (\text{i.e. } \hat{\rho} = 0), \end{aligned}\right\} \quad (2.10)$$

where ρ_0 and β are constants. In this case, (2.9) gives

$$u = \left(\frac{\beta g}{3}\right)^{\frac{1}{2}}\left(\frac{1-z^2}{1-\beta z}\right)^{\frac{1}{2}} = \left(\frac{\beta g}{6}\right)^{\frac{1}{2}}\left(\frac{4-9y^2}{2-3\beta y}\right)^{\frac{1}{2}}.$$

The velocity is thus a maximum at $y = \tfrac{1}{3}\beta + O(\beta^3)$ (i.e. just above the bottom of the channel if β is small) and falls continuously to zero as $y \to \tfrac{2}{3}$.

3. Generalization of the variational principle

It has been shown that among all horizontal steady flows realizable by withdrawing fluid from a bottom layer of a reservoir with any given density stratification, the self-similar flow described by (2.7) uniquely achieves the maximum possible flow force S. This principle will now be extended to two-dimensional wavy flows, which plainly may be realized in certain circumstances, as when the flow passes into a uniform stretch of channel downstream of an obstacle.

For any two-dimensional flow of a stratified but incompressible fluid, in which the velocity components are u and v respective to x and y, there is a stream-function ψ by virtue of the fact that div $(u, v) = 0$. Moreover, when the flow is steady, the stream-lines $\psi =$ const. coincide with the lines $\rho =$ const. and so also, in the present description, with the lines $z =$ const. We may accordingly take $y = y(x, z)$ as the dependent variable and use the fact that ψ, ρ and R are functions of z alone. The velocity components are then given by

$$u = \psi_y = \psi'(z)\frac{1}{y_z},$$

$$v = -\psi_x = \psi'(z)\frac{y_x}{y_z},$$

and it is helpful to write

$$F(z) = \rho(z)\,[\psi'(z)]^2.$$

In general this function is not prescribable *a priori*, but for all self-similar flows (see §4) it is proportional to the function of z expressed on the right-hand side of (2.9).

On the assumption that $y_z > 0$, the partial differential equation satisfied by $y(x,z)$ may be found easily enough from the Euler equations of steady motion, or even more readily by transforming the equation introduced by Long (1953) for $\psi(x,y)$, as simplified by Yih (1965, p. 76) as an equation for $\rho^{\frac{1}{2}}\psi$. It is

$$-F(z)\,[y_x/y_z]_x + \tfrac{1}{2}[F(z)\,(1+y_x^2)/y_z^2]_z + g\rho'(z)\,y - R'(z) = 0. \tag{3.1}$$

Except that the coefficient function F is undetermined, this equation resembles one that has been used previously for somewhat simpler problems of internal waves assumed to arise from a base flow with uniform horizontal velocity (cf. Benjamin 1967, equation (3.3), also Turner 1980 where an exact mathematical treatment is given). The boundary conditions to be satisfied by the solution y of (3.1) are the kinematical condition at the bottom of the moving fluid,

$$y(x,0) = 0 \quad \forall x, \tag{3.2}$$

and the dynamical condition ensuring that at the top $z = H-$, with $y(x, H-) = h(x)$, say, the pressure is $\rho_+(H-h)$. When this pressure is alternatively expressed by (2.3) with $R(H-) = g\rho_s H$, the condition is seen to take the form

$$\tfrac{1}{2}F(H)\left\{\frac{1+h_x^2}{y_z^2\,(x, H-)}\right\} = g\hat{\rho}(H-h) \quad \forall x. \tag{3.3}$$

In the case that $\hat{\rho} = 0$ and consequently $F \downarrow 0$ as $z \uparrow H$ (see example in § 5), the upper boundary condition becomes simply that y_z remain bounded in the limit.

For present purposes it is sufficient to know that any steady wavy flow must be represented by a solution of (3.1) satisfying these boundary conditions, and our inability to prescribe the function F is in fact immaterial. The flow force S is again given by (2.5), from which p can be eliminated by means of (2.3) with $q^2 = u^2 + v^2$.

The result is

$$S = \tfrac{1}{2}g\rho_+\,(H-h)^2 + \int_0^H \left\{\tfrac{1}{2}F(z)\left(\frac{1-y_x^2}{y_z}\right) + Ry_z - g\rho y y_z\right\} dz,$$

which an integration by parts with use of (2.1) and (2.3) reduces to

$$S = \tfrac{1}{2}g\rho_+ H^2 + g\hat{\rho}(Hh - \tfrac{1}{2}h^2) + \int_0^H \left\{\tfrac{1}{2}F(z)\left(\frac{1-y_x^2}{y_z}\right) - g\rho'(z)\,(zy - \tfrac{1}{2}y^2)\right\} dz. \tag{3.4}$$

By differentiation of this expression and by appeal to the fact that y satisfies (3.1) and the boundary conditions (3.2) and (3.3), it is easy to confirm that S is independent of x, as is plainly required by momentum conservation. Equation (3.1) and the boundary conditions can be used further to reduce the terms in (3.4) dependent on $F(z)$. Integrations by parts lead directly to

$$S/g = \tfrac{1}{2}\rho_+ H^2 + \hat{\rho}(2Hh - \tfrac{3}{2}h^2) + \int_0^H (-\rho')(2zy - \tfrac{3}{2}y^2)\,dz - \int_0^H F(z)\left\{\frac{y_x^2}{y_z} + y\left(\frac{y_x}{y_z}\right)_x\right\} dz, \tag{3.5}$$

which recovers (2.6) in the case of x-independent flows. Furthermore, since S is independent of x and since the average of the second integral in (3.5) is zero over a wavelength of a flow periodic in x (or, more generally, is zero between any two stations where y is the same or where $y_x = 0\,\forall\, z \in [0, H)$), we can conclude from (3.5) that for a wavy flow

$$S = \text{average of}\, S_0\,(y), \tag{3.6}$$

where $S_0\,(y)$ denotes the expression (2.6) with x-dependence of y now allowed.

In the light of the discussion following (2.6), it follows immediately from (3.6) that for every wavy flow S is less than the value S_m realized uniquely by the special flow that (2.7) describes. The promised generalization of the variational principle is thus established.

Note also that whatever the average $\bar{y}(z)$ of $y(x, z)$ with respect to x in a wavy flow, (3.6) implies that $S < S_0(\bar{y})$. This conclusion accords with a property well known from studies of other formulations of internal-wave problems, namely that when periodic waves can be superposed without energy loss on an originally horizontal flow, the process entails a reduction in flow force (cf. Benjamin 1966, §2).

4. Gradually varying flows

Let us extend the ideas of §2 to the case of steady flow from a reservoir into a channel with planform as illustrated in figure 2. The breadth b of the channel varies continuously with horizontal distance x, having a minimum value b_c as indicated in the figure, and its variation is assumed to be so gradual that the velocity u of the fluid in the x-direction is the only component significantly entering the Bernoulli law (2.3). Moreover, u is taken to be uniform across the span of the flow, although of course its value depends on height above the bottom. The theory to be developed is thus a counterpart of the shallow-water approximation for gradually varying open-channel flows of a homogeneous fluid.

Conservation of mass in the incompressible fluid can be expressed by considering an elementary stratum of vertical thickness δz in the reservoir, from which the fluid flows into a sheet whose local thickness in the channel is δy. If $\bar{u}$ is locally the mean horizontal velocity in the sheet, then

$$\bar{u}b\,\delta y = \delta Q$$

is the element of volume flux which must be independent of x when the flow is steady. Dividing by δz and taking the limit as $\delta z \to 0$, we have

$$uby' = Q', \tag{4.1}$$

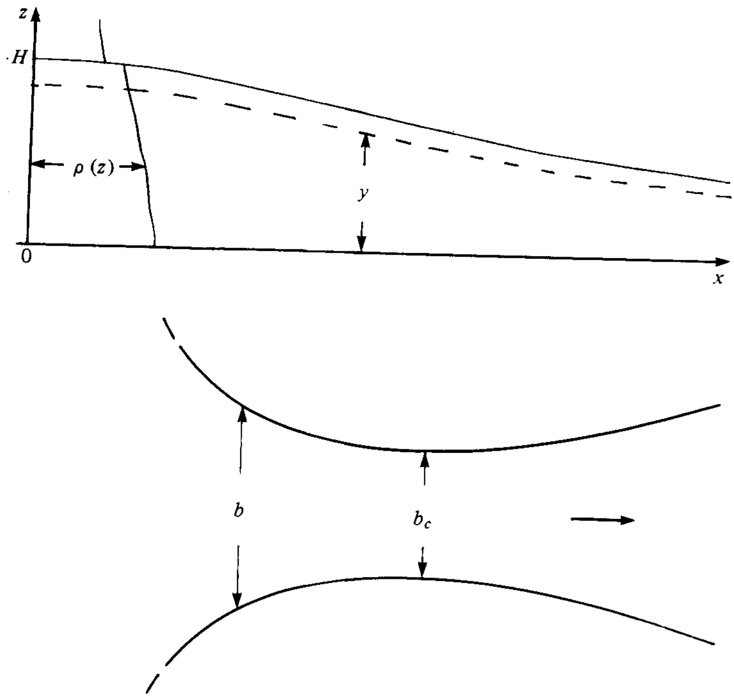

FIGURE 2. Flow along a convergent-divergent channel.

where the z-derivative Q' is evidently a function of z alone. The expression (2.4) for ρu^2 is applicable as an approximation everywhere according to the present assumptions (cf. Yih 1969), and the elimination of u between it and (4.1) gives

$$2\lambda y'^2\left\{\hat{\rho}(H-h)-\int_z^H(\xi-y)\,\rho'(\xi)\,d\xi\right\}=f(z), \tag{4.2}$$

in which $\lambda = b^2/b_c^2$, $h = y(H)$ and

$$f(z) = \rho(z)\,[Q'(z)]^2/gb_c^2.$$

This equation for $y(z)$ on $[0, H]$ is complemented by the boundary condition $y(0) = 0$, and the dependence of the flow on its position along the channel enters through the parameter λ which varies from 1 to ∞.

The problem may alternatively be expressed as a second-order differential equation with a pair of boundary conditions. Dividing (4.2) by y'^2 and differentiating with respect to z, we obtain

$$(f/y'^2)'+2\lambda(y-z)\,\rho' = 0, \tag{4.3}$$

and the boundary conditions are

$$y(0,\lambda) = 0,\; 2\lambda\hat{\rho}\{H-y(H,\lambda)\}\,[y'(H,\lambda)]^2 = f(H-). \tag{4.4}$$

This form of the problem corresponds, of course, to the x-independent version of (3.1)–(3.3) with $F(z) = gf(z)/\lambda$. As noted earlier, in the case that $\hat{\rho} = 0$ and consequently $f(H-) = 0$, the second condition in (4.4) is replaced by a condition of regularity as $z \to H$, i.e. $|y'(H-,\lambda)| < \infty$.

The non-linear parametrized problem (4.3) and (4.4) is not amenable to comprehensive treatment that allows arbitrary specification of the coefficient function f. A somewhat perplexing situation is presented, moreover, in that f may not be a prescribable feature of the physical problem. It will generally depend on conditions imposed at the downstream termination of the flow, and there appears to be no simple argument delimiting the complete class of functions f that are physically relevant. However, by analogy with the corresponding problem for open-channel flows of a homogeneous fluid, we may reasonably look to the possibility of solutions with the following properties:

(i) $y(z,\lambda)$ varies continuously with $\lambda > 1$ (hence continuously with distance x along the channel).

(ii) The solution in $1 \leqslant \lambda < \infty$ has two branches which are confluent at $\lambda = 1$ (i.e. the flow can be different at channel sections with the same breadth upstream and downstream of the minimum section).

(iii) On one branch, $y(z,\lambda) \to z$ as $\lambda \to \infty$ (i.e. the flow connects smoothly with the reservoir).

Self-similar flows

As was appreciated in §1, these special flows have previously been noticed to be solutions of the above problem. If we take

$$y = \kappa(\lambda)\, z, \tag{4.5}$$

which satisfies the boundary condition on the channel bottom, then equation (4.1) becomes

$$2\lambda(\kappa^2-\kappa^3)\left\{\hat{\rho}H - \int_z^H \xi\rho'(\xi)\,d\xi\right\} = f(z).$$

Hence (4.5) is a solution with the required properties if

$$\lambda(\kappa^2-\kappa^3) = \tfrac{4}{27}, \tag{4.6}$$

and

$$f(z) = \tfrac{8}{27}\left\{\hat{\rho}H - \int_z^H \xi\rho'(\xi)\,d\xi\right\} = \frac{8\{R(z)-R(H+)\}}{27g}. \tag{4.7}$$

The cubic (4.6) for κ has a double root $\kappa_c = \frac{2}{3}$ for $\lambda = 1$; and it has two distinct positive roots for $\lambda > 1$, one of which tends to 1 and the other to zero as $\lambda \to \infty$. Thus the properties (i), (ii) and (iii) are all provided. In fact, $h = \kappa(\lambda)\, H$ is precisely the local depth of a 'choked' flow of homogeneous heavy fluid along an open channel of the same planform $\lambda = \lambda(x)$ [i.e. the flow through a Venturi flume, which is subcritical ($u^2 < gh$) upstream and supercritical downstream from the minimum section].

It remains, of course, to demonstrate the significance of this simple result. Since other flows are generally possible from stratified reservoirs, there is a need to explain how and why self-similar flows might be generated.

Regularity of solutions

Let us examine conditions for a solution $y[z,\lambda(x)]$ of (4.3) and (4.4) to vary smoothly with distance x along the channel. Assuming $\lambda(x)$ to be continuously differentiable, differentiating (4.3) with respect to x and writing

$$\phi(x,\lambda) = \frac{\partial y}{\partial x} = \frac{d\lambda}{dx}\frac{\partial y}{\partial \lambda},$$

we obtain

$$\left(\frac{f\phi'}{y'^3}\right)' - \lambda\rho'\phi = (y-z)\,\rho'\frac{d\lambda}{dx}. \tag{4.8}$$

Similarly, the boundary conditions for ϕ are seen from (4.4) to be

$$\phi(0) = 0,\ \ y'(H)\,\phi(H) - 2(H-h)\,\phi'(H) = \frac{y'(H)\,(H-h)}{\lambda}\frac{d\lambda}{dx}. \tag{4.9}$$

(Here the dependence of ϕ and y on λ is left implicit.) For a smooth solution of the parametric problem (4.3) and (4.4), the linear problem (4.8) and (4.9) for ϕ must also have a solution for each relevant value of the parameter λ. Accordingly, although the issue is trivial in the case of self-similar flows, something about the general case may be learned by means of the Fredholm alternative principle.

Inquiries on this basis have so far made limited progress. Some comparatively easy conclusions will be noted below, but the main proposition in view can only be stated as a conjecture, made plausible by the outcome of the linearized perturbation theory to be presented in § 5. It is that except for the self-similar flow described by (4.5)–(4.7), the system (4.3) and (4.4) has no continuous solution extending to the minimum section where $\lambda = 1$, $d\lambda/dx = 0$, and being such that $\phi \not\equiv 0$ there.

At the minimum section, a possibility according to (4.8) and (4.9) is that $\phi \equiv 0$, in which case the solution $y(z, \lambda)$ on the downstream side returns along the same branch as it approaches $\lambda = 1$ on the upstream side. For reasons that will be noted in § 7, this possibility has comparatively little interest, and it will not be considered further here. Thus, at $\lambda = 1$, ϕ is required to be a non-trivial solution of the homogeneous version of the linear boundary-value problem (4.8) and (4.9).

The meaning of this requirement is made clearer by transforming the left-hand sides of (4.8) and (4.9) so that the local height y of the stream surfaces is the independent variable. Thus, writing $\phi = \eta(y)$ and using (4.2) together with (2.4), one derives

$$\left.\begin{aligned} &(\rho u^2\eta_y)_y - g\rho\eta = 0, \\ &\eta(0) = 0, \quad \rho u^2\eta_y = g\hat{\rho}\eta \quad \text{at} \quad y = h. \end{aligned}\right\} \tag{4.10}$$

The existence of a non-trivial solution of (4.10) is recognizable as being just the condition for an infinitesimal wave of extreme length to be superposable without energy loss *on a horizontal flow that has the given velocity $u = u(y)$ and density distribution $\rho = \rho(y)$* [cf. Benjamin 1966, § 3.3]. So the flow at $\lambda = 1$ is *critical* in the usual sense of the term. This result is to be expected, of course, because under the present shallow-water assumptions the condition that the flow have a smooth non-zero variation through the throat of the channel is evidently equivalent to a horizontal flow at $\lambda = 1$ admitting a long-wave perturbation.

It may straightforwardly be shown that, in all cases, the accelerating flow in the divergent part of the channel is supercritical in the sense that a free infinitesimal long wave propagating against it would be swept downstream. This property corresponds to $\lambda < \gamma_1$, where γ_1 is the lowest eigenvalue of the homogeneous Sturm–Liouville problem related to (4.8) and (4.9) [i.e. with γ in place of λ and $d\lambda/dx = 0$]. The existence of ϕ everywhere downstream of the throat is hence ensured according to the Fredholm principle.

5. Flows close to the self-similar

Again with regard to a channel of gradually varying breadth, we consider steady flows that are small perturbations from the self-similar flow defined by (4.5)–(4.7). The solution is now expressed in the form

$$y = \kappa z + \epsilon\zeta(z, \kappa) \tag{5.1}$$

where the parameter κ varies from $0+$ to 1, being related to λ by (4.6), and ϵ is an infinitesimal number. For simplicity of illustration the discussion refers particularly to the example specified in (2.10), but the means will be indicated whereby the conclusions can readily be extended to all other examples.

In the chosen example, equation (4.3) becomes

$$(\sigma/y'^2)' - 2\lambda(y-z) = 0 \tag{5.2}$$

on $0 \leqslant z \leqslant 1$, with $\sigma = f(z)/\rho_0\beta$, $\sigma(1) = 0$; and the boundary conditions are

$$y(0, \kappa) = 0, \quad |y'(1, \kappa)| < \infty. \tag{5.3}$$

In keeping with (5.1), the coefficient function $\sigma(z)$ is represented as an infinitesimal perturbation from its form given by (4.5) for the self-similar flow, thus

$$\sigma(z) = \tfrac{4}{27}(1-z^2) + \epsilon r(z). \tag{5.4}$$

The perturbed flow is assumed to originate from the same bottom layer in the reservoir, and so it is implied that $r(1) = 0$.

After substitution of (5.1) and (5.4) into (5.2), linearization in ϵ gives

$$\{(1-z^2)\zeta'\}' + \mu\zeta = \alpha r', \tag{5.5}$$

in which

$$\mu = (27/4)\lambda\kappa^3 = \kappa/(1-\kappa),$$

and

$$\alpha = (27/8)\kappa$$

have, like κ, two positive values for each $\lambda > 1$. The number μ equals 2 at the critical section ($\kappa = \frac{2}{3}$) and increases smoothly with λ on the upstream side, with $\mu \to \infty$ as $\lambda \to \infty$, $\kappa \uparrow 1$. On the downstream side, $\kappa < \frac{2}{3}$ and therefore $\mu < 2$. The required solution ζ of (5.5) must, of course, satisfy the boundary conditions (5.3).

Now, the homogeneous equation corresponding to (5.5) is Legendre's equation, which has a non-trivial solution bounded on $[-1, 1]$, the respective Legendre polynomial $P_m(z)$, when μ takes the succession of values $m(m+1)$ $(m = 1, 2, \ldots)$. In view of the first boundary condition, only the odd-order polynomials which vanish at $z = 0$ are relevant here, but they comprise a basis in $L^2(0, 1)$. Thus it is merely enough that $r' \in L^2$ for there to exist a representation

$$r' = \sum_{n=1}^{\infty} a_n P_{2n-1}(z), \quad a_n = (4n-1)\int_0^1 r' P_{2n-1}(z)\, dz, \tag{5.6}$$

in which the sequence of coefficients $\{a_n\} \in l^2$. Accordingly, the formal solution of (5.5) satisfying the boundary conditions is

$$\zeta = \sum_{n=1}^{\infty} \frac{\alpha}{\mu - 2n(2n-1)} a_n P_{2n-1}(z), \tag{5.7}$$

which, since α is a bounded positive number, is meaningful *except* where the denominator $\mu - 2n(2n-1)$ vanishes for any coefficient with $a_n \neq 0$. [Note incidentally that, with these exceptions, the attribution $r' \in L^2$ implies the existence of at least a weak solution $\zeta \in H^1(0,1)$, and further regularity of ζ follows from that or r. For example, if $r \in C^1$, then $\zeta \in C^2$ and the ordinary differential equation (5.5) is satisfied pointwise.]

The first conclusion to be drawn from (5.7) is that if the perturbed flow is to exist at a throat in the channel, where $\mu = 2$, then $a_1 = 0$ in the expansion (5.6). On the downstream side, we have that $\mu - 2n(2n-1) < 0$ for all $n = 1, 2, \ldots$, and thus the solution (5.7) remains meaningful everywhere in the supercritical region. On the upstream side, however, every one of the numbers $2n(2n-1) \geqslant 12$ is crossed by μ, and so (5.7) is valid everywhere only if $a_n = 0$ for all n. The completeness of the basis $\{P_{2n-1}\}$ hence establishes the significant conclusion that *no smooth solution of the shallow-water equations exists neighbouring on the self-similar flow from the given reservoir.* Every perturbed steady flow suffers at least one local crisis where the shallow-water approximation ceases to be valid, however gradual the variation in breadth along the channel.

This conclusion readily extends to all other examples of stable density stratification in the reservoir. For each, in place of (5.5), an equation with another Sturm–Liouville operator on the left-hand side will be posed, and if $\hat{\rho} > 0$ the upper boundary condition will be given by linearizing the second of (4.4). By the Riesz–Fischer theorem, the set of eigensolutions for each respective Sturm–Liouville problem is complete as a basis in $L^2(0, H)$, and accordingly the argument proceeds as above.

It deserves emphasis that a failure of the shallow-water approximation in some part of the flow does not necessarily invalidate the approximate solution (5.7) elsewhere. The nature of the local crises that all flows neighbouring the self-similar have been shown to suffer will not be explored in any detail here, although it is an interesting matter that should be worth further study. The phenomena indicated are presumably continuous processes according to a more accurate perfect-fluid model, but to describe them one needs to abandon the hydrostatic approximation for pressure and use explicitly x-dependent differential equations such as (3.1). A particular component with $n > 1$ in the Sturm–Liouville expansion of $\zeta(z, \kappa)$ will be better modelled in the vicinity of its crisis point, say $x = 0$, by a function of the form $w_n(x)\xi_n(z)$, where $\xi_n(x)$ is the eigenfunction in question (e.g. $P_{2n-1}(z)$ above) and where, to a first approximation, a suitably normalized $w_n(x)$ will satisfy the equation

$$k^{-2}(d^2 w_n/dx^2) - x\, w_n = 1,$$

in which k^{-2} is a positive parameter proportional to $-d\mu/dx > 0$ at $x = 0$. The specification $k^{-2} = 0$ recovers the unacceptable local singularity given by the shallow-water approximation. But with $k^{-2} > 0$ this equation has a solution that is bounded on $-\infty < x < \infty$. Expressible in terms of Airy functions, the needed solution is $-\pi Gi(kx)$ in the notation used by Abramowitz & Stegun (1965, p. 448). For large $kx > 0$, the solution is quickly asymptotic to $-x^{-1}$; and for large $kx < 0$,

$$w_n \sim \pi^{\frac{1}{2}}(-kx)^{-\frac{1}{4}} \cos\left[\tfrac{2}{3}(-kx)^{\frac{3}{2}} + \tfrac{1}{4}\pi\right] - x^{-1} + O[(-kx)^{-\frac{7}{4}}].$$

The slowly diminishing but increasingly rapid oscillations may roughly simulate what in fact happens upstream of a crisis point.

6. The extremal property of supercritical self-similar flows

We return to the flow-force principle demonstrated in § 2, now considering it subject to the constraint that the flows competing for the maximum of S have passed through the throat into the divergent part of the channel and so have become supercritical. Let subscript 1 refer to any particular station downstream (i.e. $\lambda_1 > 1$, $\kappa_1 < \frac{2}{3}$), and let S_{1m} denote the value of S given by substitution of $y = \kappa_1 z$ in (2.6). Taking the expression (5.1) for y but no longer assuming ϵ to be infinitesimal, we obtain from (2.6)

$$S_1/g = S_{1m}/g + \epsilon(2-3\kappa_1)\left\{\hat{\rho}H\zeta_1(H) + \int_0^H (-\rho')\, z\zeta_1\, dz\right\} \\ -\tfrac{3}{2}\epsilon^2\left\{\hat{\rho}\zeta_1^2(H) + \int_0^H (-\rho')\,\zeta_1^2\, dz\right\}. \qquad (6.1)$$

Here the coefficient of ϵ is to be treated as the first variation $\dot{S}_1/g$ of S_1/g among flows differing from the self-similar flow.

On the other hand, when (5.1) is substituted and (4.6) is used, (4.2) leads to

$$\Gamma = \int_0^H \tfrac{1}{2}f(z)\,dz = (4/27)\left\{\hat{\rho}H^2 - \int_0^H \rho' z^2\, dz\right\} + \epsilon\lambda\kappa(2-3\kappa)\left\{\hat{\rho}H\zeta(H,\kappa) - \int_0^H \rho'\zeta z\, dz\right\} \\ + \epsilon^2\lambda\left[(1-\kappa)\int_0^H \zeta'^2\left\{\hat{\rho}H - \int_z^H \rho'(\xi)\,\xi\, d\xi\right\}dz \right. \\ \left. - 2\kappa\left\{\hat{\rho}\zeta^2(H,\kappa) - \int_0^H \rho'\zeta^2\, dz\right\}\right] + O(\epsilon^3), \qquad (6.2)$$

which by the definition of $f(z)$ must be the same for all κ such that $y(z,\kappa)$ exists. Note that whereas local failures of the shallow-water approximation may occur upstream of the throat, (6.2) still holds everywhere downstream. Writing (6.2) as $\Gamma = \Gamma_0 + \epsilon\dot{\Gamma} + \frac{1}{2}\epsilon^2\ddot{\Gamma} + \ldots$, we have that $\dot{\Gamma} = 0$ at the throat where $\kappa = \frac{2}{3}$ and therefore, in the limit $\epsilon \to 0$, also $\dot{\Gamma} = 0$ everywhere downstream where $0 < \kappa < \frac{2}{3}$. Since $\lambda_1\kappa_1\dot{S}_1 = g\dot{\Gamma}_1$ according to (6.1) and (6.2), it follows that

$$\dot{S}_1 = 0. \qquad (6.3)$$

Thus S_{1m} is a stationary value for variations in the class of flows that are smooth at and downstream of the throat.

The study of second and higher variations is more complicated and will merely be outlined here. One proceeds by using (6.2), in the form of the identity

$$\epsilon\dot{\Gamma}(\kappa_1) = \tfrac{1}{2}\epsilon^2\{\ddot{\Gamma}(\tfrac{2}{3}) - \ddot{\Gamma}(\kappa_1)\} + O(\epsilon^3),$$

to reduce the ϵ-term in (6.1). The results of the linearized theory summarized in § 5 are then used to evaluate $\ddot{S}_1$. It is thus shown without much difficulty that $\ddot{S}_1 < 0$. Further estimates of $S_1 - S_{1m}$ finally confirm that S_{1m} is a maximum.

7. Physical conclusions

Even within the context of perfect-fluid theory, the general problem of selective withdrawal from a stratified reservoir is largely intractable because of the freedom evidently available in posing the downstream conditions that determine the flow.

While falling far short of a general solution, the preceding results nevertheless illuminate various apparently central aspects of the problem, and on the basis of them a number of plausible interpretations can be made as follows about practical possibilities.

(1) First take the case of stratified fluid drawn steadily into a straight channel, as illustrated in figure 1, and suppose that the flow is caused by the extraction of fluid through a slot near the bottom at the end of the channel. If there is a discontinuity of density at $z = H$ (i.e. $\hat{\rho} > 0$), it is intuitively clear that the fluid above the interface will not be drawn into the slot when the extraction rate is sufficiently small, and so a definite question arises about the limiting condition beyond which the fluid originally above $z = H$ in the reservoir begins to be extracted. In other words, what is the 'drawdown condition' corresponding to given $H, \hat{\rho}$ and the function $\rho(z)$ on $[0, H)$?

According to the basic notation recalled at the end of § 1, a progressively larger flow force S must be manifested in the channel as the process of extraction causing the flow is intensified. But the results of §§ 2 and 3 show that the maximum possible S achievable without the entrainment of fluid originally above $z = H$ is realized by the critical self-similar flow with $y = \frac{2}{3}z$. This flow therefore comprises an upper limit for the possible drawdown condition. If drawdown does not occur until increased extraction raises S to S_m, then this flow is necessarily realized at the limit, and no further increase of S is possible without drawdown.

(2) The principle of maximum flow force was demonstrated in §§ 2 and 3 without regard to the grading of the contraction through which the flow approaches the channel, but this factor evidently will determine whether the limiting flow is realizable. It is plausible that this flow does precede drawdown when the contraction is extremely gradual. Otherwise, as the known behaviour of open-channel flows suggests, an approach to the critical condition is liable to be interrupted by wave formation. [Note that in practice it is found difficult to produce a smooth open-channel flow in a slightly subcritical condition, say with $0.6 < F < 1$, where $F = u/(gh)^{\frac{1}{2}}$ (cf. Binnie *et al.* 1955).]

A wavy flow in the channel itself is a possible precursor of drawdown when the contraction is not gradual enough, but other possibilities are indicated by the findings of § 5. Suppose that the contraction is very gradual but the sink of fluid at the end of the channel is so arranged that development of a self-similar flow is hindered (e.g. there are two slots extracting fluid). Then as S is raised towards the value at which drawdown begins, the flow may still differ appreciably from the self-similar flow, and consequently a local crisis giving rise to waves may occur some way upstream.

(3) It must be acknowledged that the present estimate of the drawdown condition disagrees radically with the view of the matter proposed by Huber (1960), who calculated a critical condition at which the flow of a stratum of homogeneous perfect fluid towards a line sink first entrains a superposed layer of fluid with smaller density. The present interpretation also conflicts in principle, but is in its outcome more easily reconciled, with a calculation by Craya (1949) on a quite different, approximate basis. The two estimates were discussed by Yih (1965, p. 128), who cited unpublished experimental results showing a large discrepancy with Huber's theoretical prediction. In terms of the Froude number F for the flow in the lower stratum (with the density difference incorporated into F in the usual way), the drawdown condition was calculated to be $F = 1{\cdot}66$, whereas according to present ideas it is just $F = 1$. The former value can at once be rejected as a practical threshold for flows originating from a large

reservoir, because such flows with $F > 1$ and stagnant fluid above are impossible in the absence of a throat upstream – a feature not recognized in Huber's model. The experimental drawdown condition reported by Yih is roughly $F = 0{\cdot}7$, which is quite consistent with the present interpretation when allowance is made as suggested for the possibility of wave formation forestalling the idealized critical condition $F = 1$.

(4) Cases where ρ is continuous at $z = H$ provide a different interpretation. Note that whatever the flow force determined by the process of extraction at the end of the channel, there is a least value of H for which the required S can be realized. Then $S = S_m(H)$, where S_m is the maximum of S for a given H, as given by (2.8). In other words, for a given flow force, either a flow is developed having $y = \frac{2}{3}z$ in $[0, H]$ for the respective minimum H, or a deeper layer of fluid is drawn into motion. At least in a straight channel following a gradual contraction, it may therefore be expected that self-similar flows will tend to be realized whenever fluid is withdrawn from a continuously stratified reservoir.

(5) As is known for open-channel flows of a homogeneous fluid, steady stratified-fluid flows along a convergent-divergent channel are likely to be in better accord with shallow-water theory than near-critical flows in a straight channel. An everywhere subcritical flow with $\partial y/\partial x = 0$ at the minimum section is a theoretical possibility; but, as the open-channel analogy shows, it is liable to be swept away in the divergent part of the channel unless the final outflow is specially restricted. The raising of downstream flow force concomitantly with increasing the extraction rate will generally produce a flow that is critical at the throat and supercritical downstream. The principle demonstrated in § 6 accordingly indicates that a self-similar flow will tend to be established in this situation, since it gives rise, at the downstream end of the channel, to a flow force that is the maximum possible without additional fluid being extracted. This conclusion must be regarded with caution, however, in view of the artificial feature that a layer of stagnant fluid deeper than $\frac{1}{3}H$ lies above the fast, supercritical flow. The possibility that instabilities of the Kelvin-Helmholtz type may precipitate a dissipative transition (hydraulic jump) back to subcritical flow, also the possibility of a super-critical drawdown condition such as found by Huber (1960), suggests that the flows in question may not be realizable in a far supercritical condition.

It is noteworthy that Wood (1968), considering a two-layer model, derived a self-similar flow in a convergent-divergent channel on the basis of the hypothesis that the shallow-water equations should have a smooth solution everywhere. His result agrees, of course, with the observations made in §§ 4 and 5, but the present flow-force principle, rather than an arbitrary hypothesis of smoothness, is much more telling as a reason why the self-similar flow should be generated. Wood also presented some experimental results approximately confirming his prediction.

(6) The present investigation has focussed on the case of flow in layers lying on a horizontal plane, but all aspects of the theory extend more or less immediately to the case of withdrawal from internal layers. The first boundary condition in (4.4) has to be replaced by another, akin to the second of (4.4), applying at the lower interface with stagnant fluid, and the self-similar solution corresponding to (4.5)–(4.7), but now defined on $[-H, H]$, say, is appropriately modified. The example considered in § 2, for instance, extends precisely to a solution like (5.1) on $[-1, 1]$, and the even as well as odd Legendre polynomials are then required to express an arbitrary perturbation in the manner of (5.6) and (5.7).

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